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# LOCAL DIMENSION-FREE ESTIMATES FOR VOLUMES OF SUBLEVEL SETS OF ANALYTIC FUNCTIONS*

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ABSTRACT

We derive sufficiently sharp local dimension-free estimates for volumes of sublevel sets of analytic functions in the unit ball of $\mathbb{C}^n$.

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## 1. Introduction and result

Let $F$ be a non-constant real-analytic function in the unit ball in $\mathbb{R}^n$. We are interested here in dimensionless estimates of the size of sub- and super-level sets $\{x : |F(x)| \lesseqgtr t\}$. To simplify the problem and avoid dependence on the domain of analyticity of $F$, we assume that $F$ is analytic in the complex unit ball in $\mathbb{C}^n$.

We denote complex balls $\{z \in \mathbb{C}^n : |z - w| < r\}$ by $B_c(w, r)$, and real balls $\{x \in \mathbb{R}^n : |x - u| < r\}$ by $B(u, r)$. For any real ball $B$, we denote by $\mathrm{Vol}_B$ the normalized volume

$$\mathrm{Vol}_B(E) = \frac{\mathrm{Vol}(B \cap E)}{\mathrm{Vol}(B)}.$$

Let $F$ be a non-constant analytic function in $B_c(0, 1)$, and let $B \subset B(0, 1 - \varepsilon)$ be a real ball. We look for the upper bounds for the distribution functions $t \mapsto \mathrm{Vol}_B\{|F| \leqslant tM_B(F)\}$ $(t < 1)$, and $t \mapsto \mathrm{Vol}_B\{|F| \geqslant tM_B(F)\}$ $(t > 1)$. The quantity $M_B(F)$ normalizes the distribution function of $|F|$ in $B$. We would like to keep our estimates dimensionless and universal: their right-hand sides should depend on a global "degree" $d_F$ of $F$ in the complex unit ball, and on the distance $\varepsilon$ from the ball $B$ to the unit sphere, but should not depend on the number of variables $n$, and on the choice of the ball $B$.

The choice of the degree

$$d_F = \log \frac{\sup_{B_c(0,1)} |F|}{|F(0)|}$$

is suggested by the one-dimensional case when local bounds follow from the classical Cartan lemma [L]. A traditional statistical normalization of the distribution function uses the median, that is, a number $m_B(F)$ such that

$$\mathrm{Vol}_B\{|F| \geqslant m_B(F)\} = \frac{1}{2}.$$

To make the constants simpler, we choose for the normalization the $e^{-1}$-quantile, that is, a number $M_B(F)$ such that

$$\mathrm{Vol}_B\{|F| \geqslant M_B(F)\} = 1/e.$$

Since $|F|$ is a real-analytic function, the quantile $M_B(F)$ is uniquely defined if $|F|$ is a non-constant function in $B(0, 1)$.

Our main result is

THEOREM: *Let $F$ be a non-constant analytic function in the unit ball $B_c(0, 1)$, and let $B$ be any real ball contained in $B(0, 1 - \varepsilon)$, $\varepsilon \leqslant \frac{1}{4}$. Then, for every $\lambda > 1$,*

$$\mathrm{Vol}_B\{|F| \leqslant (C\lambda)^{-\sigma} M_B(F)\} \leqslant 1/\lambda, \tag{1.1}$$

*and*

$$\mathrm{Vol}_B\{|F| \geqslant (C\lambda)^\sigma M_B(F)\} \leqslant e^{-\lambda}, \tag{1.2}$$

*where one can take* $C = 64$, *and* $\sigma = \frac{48}{\varepsilon^3} \cdot d_F$.

*Remark:* There is another popular choice of the normalization constant $M_B(F)$ as the $L^q(B)$-norm of $F$ for some $q \geqslant 0$. Having estimates (1.1) and (1.2), it is not difficult to check that for every positive $q$

$$(eC)^{-\sigma} M_B(F) \leqslant ||F||_{L^0(B)} \leqslant ||F||_{L^q(B)} \leqslant (3C \max\{1, q\sigma\})^\sigma M_B(F)$$

(cf. [NSV, §2]). Hence one can choose any of the $L^q(B)$-norms of $F$ $(0 \leqslant q < \infty)$ for the normalization at the cost of more clumsy expressions on the righthand sides of (1.1) and (1.2). Simple examples show that one cannot use the $L^\infty(B)$-norm for the normalization and keep the result dimension-free.

The theorem appeared as an attempt to "generalize" a similar statement for polynomials $P$ in $\mathbb{R}^n$ which says that the counterparts of (1.1) and (1.2) hold with $\sigma = \deg P$ and $C = 4$ in *any* convex body $V \subset \mathbb{R}^n$ (see [NSV]). The difference between the usual degree of polynomials and our degree $d_F$ is that $d_F$ can blow up when we restrict $F$ to one-dimensional disks in $B_c(0,1)$. For this reason, we cannot apply directly to $F$ the needle decomposition [GM2, LS] which reduces dimensionless estimates to uniform estimates for all one-dimensional traces of $F$. First, we need to perform a change of variables in $F$ (see Lemma C below).

For the same reason, only balls $B$ appear in the assertion of the theorem instead of arbitrary convex bodies $V$. Without changing the proof, one can replace real balls $B$ by convex bodies $V \subset B(0, 1-\varepsilon)$ whose boundaries have sectional curvatures bounded from below by some fixed positive constant. This "curvature" restriction can, probably, be relaxed but cannot be removed completely: a simple example given at the end of this note shows that estimates (1.1) and (1.2) may fail for thin two-dimensional rectangles.

If needed, the reader can adjust the theorem to plurisubharmonic functions in the unit ball of $\mathbb{C}^n$ and to analytic functions with values in a Banach space. This requires only minor changes in Lemma B below (cf. [CW]).

Compiling the theorem with the technique from [NSV, §3], one can obtain an Offord-type statement about the distribution of zeroes of analytic functions in families that depend analytically on some parameters. Informally speaking, the result is that the portion of the family occupied by the functions whose distribution of zeroes deviates from the "average" one by some fixed amount, is about Const exp{−size of the deviation}. This might be a possible embryo of a non-linear and dimensionless value-distribution theory.

There are many publications with *dimension-dependent* versions of the theorem. For polynomials, the first results were obtained by R. M. Dudley and B. Randol [DR] and Yu. Brudnyi and M. Ganzburg [BG]. For real- and complex-analytic functions, various results still depending on the dimension $n$ are due to N. Nadirashvili [N1, N2], N. Garofalo and P. Garrett [GG], and A. Brudnyi [Br1], [Br2]. By no means is this list complete.

As to the history of dimensionless estimates (1.1) and (1.2), the pioneering dimensionless results are due to A. C. Offord [O], M. Gromov and V. Milman [GM1] (the case of linear functions), and J. Bourgain [B] (a cruder form of (1.2) for polynomials). For other developments, see A. Brudnyi [Br2, Theorem 1.11], S. Bobkov [Bo], and A. Carbery and J. Wright [CW].

## 2. Proof of the Theorem

The proof of the theorem will be prepared from three ingredients.

A. The geometric Kannan–Lovász–Simonovits lemma. A continuous function $\Phi\colon \mathbb{R}^n \to \mathbb{R}_+$ is called **logarithmically concave** if

$$\Phi\left(\frac{x+y}{2}\right) \geqslant \sqrt{\Phi(x)\Phi(y)}$$

for all $x, y \in \mathbb{R}^n$.

Lemma A: *Let $\Phi$ be a logarithmically concave function in $\mathbb{R}^n$. Let $S \subset \operatorname{supp}(\Phi)$ be a convex compact, and let $E \subset S$ be a closed subset. For $\lambda > 1$, define*

$$E_{\lambda,S} := \left\{x \in E : \frac{|E \cap J|}{|J|} \geqslant \frac{\lambda-1}{\lambda} \text{ for every interval } J \text{ such that } x \in J \subset S\right\}.$$

*Then*

$$\frac{\int_{E_{\lambda,S}} \Phi}{\int_S \Phi} \leqslant \left(\frac{\int_E \Phi}{\int_S \Phi}\right)^{\lambda}.$$

This lemma was proved in [NSV] using the needle decomposition (= convex partition) technique developed by M. Gromov and V. Milman [GM2] and by L. Lovász and M. Simonovits [LS]. It can also be derived from a result of R. Kannan, L. Lovász and M. Simonovits [KLS, Theorem 2.7].

B. One-dimensional Remez property. We shall use the following result (which, probably, should be called the Boutroux–Bloch–Cartan–Remez property):

LEMMA B: *Let $f$ be an analytic function in the unit disk $\mathbb{D}$ such that $\sup_{\mathbb{D}} |f| \leqslant 1$. Then for every interval $I \subset [-a, a]$ and every set $E \subset I$,*

$$\max_I |f| \leqslant \left(\frac{C|I|}{|E|}\right)^{\sigma} \sup_E |f|$$

*with*

$$C = 64 \quad \text{and} \quad \sigma = \frac{3}{1-a^2} \cdot \log \frac{1}{|f(a)f(-a)|}.$$

We shall prove Lemma B in §3.

C. CHANGE OF VARIABLE. Let $\delta \leqslant \frac{1}{8}$. Set $A = 1 - \delta^3$, $a = \sqrt{A}$, $\varphi(\zeta) = (A - \zeta)/(1 - A\zeta)$, and consider the mapping $T$ defined on the unit ball $B_c(0, 1)$ in $\mathbb{C}^n$ by the formula

$$T(z) := \varphi\Big(\sum_{j=1}^{n} z_j^2\Big) z, \quad z = (z_1, \ldots, z_n) \in B_c(0, 1) \subset \mathbb{C}^n.$$

In particular, $T(x) = \varphi(|x|^2)x$ for $x \in B(0, 1)$.

LEMMA C: *Set $R_0 = 1 - 3\delta - \delta^3$, $r_0 = \sqrt{R_0}$. Then the mapping $T$ has the following properties:*

(1) *$TB_c(0, 1) \subset B_c(0, 1)$;*
(2) *$T$ maps the real sphere $|x| = a$ to the origin;*
(3) *$T$ is one-to-one in the ball $B(0, r_0)$;*
(4) *$TB(0, r_0)$ is a ball centered at the origin of radius greater than $1 - 2\delta$;*
(5) *The Jacobian $|\det D_x T|$ is a logarithmically concave function in $B(0, r_0)$;*
(6) *The (partial) pre-image $B(0, r_0) \cap T^{-1}B$ of every (real) ball $B \subset TB(0, r_0)$ is convex.*

The first two properties are obvious; the others will be proved in §4.

*Proof of Theorem:* Let $F$ be a non-constant analytic function in $B_c(0, 1)$. Without loss of generality, we may assume that $|F|$ is not a constant function in $B(0, 1)$, and $\sup_{B_c(0,1)} |F| \leqslant 1$. We shall show that for every $c > 0$ and every $\lambda > 1$,

$$\mathrm{Vol}_B\{|F| \geqslant (C\lambda)^{\sigma} c\} \leqslant (\mathrm{Vol}_B\{|F| \geqslant c\})^{\lambda}. \tag{2.1}$$

The rest is the same as in [NSV]: to get (1.2), we just set $c = M_B(F)$ in (2.1); to get (1.1), we rewrite (2.1) in the form

$$\mathrm{Vol}_B\{|F| \geqslant c\} \leqslant (1 - \mathrm{Vol}_B\{|F| < (C\lambda)^{-\sigma} c\})^{\lambda}$$

and, taking $c = M_B(F)$, obtain

$$\mathrm{Vol}_B\{|F| < (C\lambda)^{-\sigma} M_B(F)\} \leqslant 1 - e^{-1/\lambda} < 1/\lambda,$$

which is identical to (1.1) since, due to the real-analyticity of $|F|^2$ in $B(0,1)$, the level sets of $|F|$ have zero volume: $\mathrm{Vol}_B\{|F| = \text{const}\} = 0$.

To prove (2.1), choose $\delta = \varepsilon/2$ and consider the composition $F_T(z) = (F \circ T)(z)$ of the function $F$ with the mapping $T$ defined above. The function $F_T$ is analytic in the complex unit ball and $\sup_{B_c(0,1)} |F_T| \leqslant 1$. The advantage we gain from this trick is that the new function $F_T$ has a lower bound on a massive set (the real sphere) instead of just one point (the origin): $F_T(u) = F(0)$ for every $u \in \mathbb{R}^n$ with $|u| = a$. Let $S = B(0, r_0) \cap T^{-1}B$. Due to Lemma C (property (6)), this is a convex compact subset of $B(0, r_0)$. We shall show that for every $c > 0$ and for every $\lambda > 1$,

$$\frac{\int_{S\cap\{|F_T|\geqslant (C\lambda)^\sigma c\}} |\det D_x T|}{\int_S |\det D_x T|} \leqslant \left( \frac{\int_{S\cap\{|F_T|\geqslant c\}} |\det D_x T|}{\int_S |\det D_x T|} \right)^\lambda \tag{2.2}$$

which is equivalent to (2.1).

Let $E = \{x \in S : |F_T(x)| \geqslant c\}$. To prove (2.2), we check that

$$S \cap \{|F_T| > (C\lambda)^\sigma c\} \subset E_{\lambda,S}, \tag{2.3}$$

where the set $E_{\lambda,S}$ is defined in Lemma A. Then Lemma A with the function $\Phi = |\det D_x T|$ (which is logarithmically concave due to property (5) in Lemma C) gives us (2.2). Here we use again that the level sets of $|F|$ have zero volume.

Assume that $x \notin E_{\lambda,S}$, i.e., that there exists an interval $J \subset S$ containing the point $x$ and such that the length of the set $J \setminus E$ is at least $\lambda^{-1}|J|$. Extend this interval until the endpoints appear on the unit sphere $\partial B(0,1)$ and denote the extended interval by $J^*$. Let $\Delta$ be the one-dimensional complex disk with diameter $J^*$. Then $\Delta \subset B_c(0,1)$ and $|F_T(x)| = |F(0)|$ for $x \in J^* \cap \partial B(0,a)$. Further, $|J^* \cap B(0,a)| \leqslant a|J^*|$ and we can apply the one-dimensional Remez property (Lemma B) to the analytic function $F_T\big|_\Delta$, the interval $J$, and its subset $J \setminus E$. We get

$$|F_T(x)| \leqslant \max_J |F_T| \leqslant \left(\frac{C|J|}{|J \setminus E|}\right)^\sigma \sup_{J \setminus E} |F_T| \leqslant (C\lambda)^\sigma c,$$

with $C = 64$, and

$$\sigma = \frac{3}{1-a^2} \cdot \log \frac{1}{|F_T(a)|^2} = \frac{6}{\delta^3} \cdot \log \frac{1}{|F(0)|} = \frac{48}{\varepsilon^3} \cdot \log \frac{1}{|F(0)|},$$

completing the proof of (2.3) and, thereby, of the theorem. ∎

## 3. Proof of Lemma B

We shall use the standard factorization $f(z) = U(z)B(z)$, where $U(z)$ has no zeroes in the disk and $B(z)$ is the Blaschke product. Since for every $x \in [-a, a]$,

$$\log|U(x)| = -\int_{\mathbb{T}} \frac{1-x^2}{|1-x\zeta|^2} d\mu(\zeta)$$

where $\mu$ is some positive measure on the unit circle $\mathbb{T}$, and since

$$\frac{1}{|1-x\zeta|^2} \leqslant \frac{1}{|1-a\zeta|^2} + \frac{1}{|1+a\zeta|^2} \tag{3.1}$$

for every $\zeta \in \mathbb{D}$, $x \in [-a, a]$ (for example, by inspection of the triangle with vertices at 1, $-a\zeta$, and $a\zeta$), we immediately conclude that

$$\begin{aligned} \log|U(x)| &\geqslant -\frac{1-x^2}{1-a^2} \int_{\mathbb{T}} \Big( \frac{1-a^2}{|1-a\zeta|^2} + \frac{1-a^2}{|1-a\zeta|^2} \Big) d\mu(\zeta) \\ &= \frac{1-x^2}{1-a^2} \log|U(a)U(-a)| \end{aligned}$$

and, therefore,

$$\min_{[-a,a]} |U| \geqslant \big|U(-a)U(a)\big|^{1/(1-a^2)}. \tag{3.2}$$

We shall split the zero set $\mathcal{Z}(f)$ of $f$ into two parts:

$$\mathcal{Z}_1(f) = \Big\{ \zeta \in \mathcal{Z}(f) : \max_{x\in[-a,a]} \frac{(1-|\zeta|^2)(1-x^2)}{|1-x\zeta|^2} \leqslant \frac{2}{3} \Big\}$$

and

$$\mathcal{Z}_2(f) = \Big\{ \zeta \in \mathcal{Z}(f) : \max_{x\in[-a,a]} \frac{(1-|\zeta|^2)(1-x^2)}{|1-x\zeta|^2} > \frac{2}{3} \Big\}.$$

Let $B(z) = B_1(z)B_2(z)$ be the corresponding decomposition of the Blaschke product $B$. Our next aim will be to show that for all $x \in [-a, a]$,

$$|B_1(x)| \geqslant \big|B_1(-a)B_1(a)\big|^{2(1-x^2)/(1-a^2)},$$

which yields

$$\min_{x\in[-a,a]} |B_1(x)| \geqslant |B_1(a)B_1(-a)|^{2/(1-a^2)}. \tag{3.3}$$

Clearly, it is enough to establish this inequality for every Blaschke factor in $B_1(z)$. Using first the inequality $1-t \geqslant e^{-2t}$ $(0 \leqslant t \leqslant \frac{2}{3})$, and then applying (3.1), we obtain

$$\begin{aligned}
\left|\frac{x-\zeta}{1-x\bar{\zeta}}\right|^2 &= 1 - \frac{(1-x^2)(1-|\zeta|^2)}{|1-x\bar{\zeta}|^2} \\
&\geqslant \exp\Big\{-2\frac{(1-x^2)(1-|\zeta|^2)}{|1-x\bar{\zeta}|^2}\Big\} \\
&\geqslant \exp\Big\{-\frac{2(1-x^2)}{1-a^2}\Big[\frac{(1-a^2)(1-|\zeta|^2)}{|1+a\bar{\zeta}|^2} + \frac{(1-a^2)(1-|\zeta|^2)}{|1-a\bar{\zeta}|^2}\Big]\Big\} \\
&\geqslant \Big\{\Big[1 - \frac{(1-a^2)(1-|\zeta|^2)}{|1+a\bar{\zeta}|^2}\Big]\cdot\Big[1 - \frac{(1-a^2)(1-|\zeta|^2)}{|1-a\bar{\zeta}|^2}\Big]\Big\}^{2(1-x^2)/(1-a^2)} \\
&= \left|\frac{-a-\zeta}{1+a\bar{\zeta}}\cdot\frac{a-\zeta}{1-a\bar{\zeta}}\right|^{2(1-x^2)/(1-a^2)},
\end{aligned}$$

proving the statement.

The next observation is that the number $N = \operatorname{card} \mathcal{Z}_2(f)$ satisfies the inequality

$$N \leqslant \frac{3}{1-a^2}\log\frac{1}{|B_2(-a)B_2(a)|} \leqslant \frac{3}{1-a^2}\log\frac{1}{|f(-a)f(a)|} = \sigma. \tag{3.4}$$

Indeed, for every zero $\zeta$ in $\mathcal{Z}_2(f)$, we have

$$\begin{aligned}
\left|\frac{-a-\zeta}{1+a\bar{\zeta}}\cdot\frac{a-\zeta}{1-a\bar{\zeta}}\right|^2 &= \Big[1 - \frac{(1-a^2)(1-|\zeta|^2)}{|1+a\bar{\zeta}|^2}\Big]\cdot\Big[1 - \frac{(1-a^2)(1-|\zeta|^2)}{|1-a\bar{\zeta}|^2}\Big] \\
&\leqslant \exp\Big\{-(1-a^2)\Big[\frac{1-|\zeta|^2}{|1+a\bar{\zeta}|^2} + \frac{1-|\zeta|^2}{|1-a\bar{\zeta}|^2}\Big]\Big\} \\
&< \exp\Big\{-(1-a^2)\frac{1-|\zeta|^2}{|1-x\zeta|^2}\Big\} \\
&\leqslant \exp\Big\{-\frac{2}{3}\frac{1-a^2}{1-x^2}\Big\} \\
&\leqslant e^{-2(1-a^2)/3}.
\end{aligned}$$

Thus,

$$|B_2(a)B_2(-a)| \leqslant \exp\Big[-\frac{N(1-a^2)}{3}\Big],$$

which yields (3.4).

Now write $B_2(z) = P(z)R(z)$, where $P(z) = \prod_{k=1}^{N}(z-\zeta_k)$ and $R(z) = \prod_{k=1}^{N} 1/(1-z\bar{\zeta}_k)$. We have

$$\max_{[-a.a]}|R| \leqslant \Big\{\prod_{\zeta\in\mathcal{Z}_2(f)} Q(\zeta)\Big\}\min_{[-a,a]}|R|, \tag{3.5}$$

where

$$Q(\zeta)=\frac{\max_{x\in[-a,a]}|1-x\zeta|}{\min_{x\in[-a,a]}|1-x\zeta|}\leqslant\frac{2}{1-|\zeta|}\leqslant\frac{4}{1-|\zeta|^2}. \tag{3.6}$$

Next, choose $x^*\in[-a,a]$ in such a way that

$$\frac{(1-|\zeta|^2)(1-x^{*2})}{|1-x^*\zeta|^2}\geqslant\frac{2}{3}.$$

This is possible since $\zeta\in\mathcal{Z}_2(f)$. We have

$$\frac{4}{1-|\zeta|^2}\leqslant\frac{3}{2}\cdot\frac{4(1-x^{*2})}{|1-x^*\zeta|^2}\leqslant 6\cdot\frac{1-x^{*2}}{(1-|x^*|)^2}=6\cdot\frac{1+|x^*|}{1-|x^*|}\leqslant\frac{12}{1-|x^*|}. \tag{3.7}$$

Further, we need an elementary inequality

$$\frac{12}{1-s}\leqslant\exp\Big\{\frac{8}{3(1-s^2)}\Big\},\quad 0\leqslant s<1, \tag{3.8}$$

which is verified by a straightforward argument. Indeed, consider the function

$$H(s)=\frac{8}{3(1-s^2)}+\log(1-s),\quad 0\leqslant s<1.$$

Then

$$H'(s)=\frac{16s}{3(1-s^2)^2}-\frac{1}{1-s}=\frac{3s^3+3s^2+13s-3}{3(1-s^2)^2}.$$

The cubic polynomial in the numerator has one real zero $s^*$ which is located in the interval $(\frac{1}{5},\frac{1}{4})$. Therefore, $H$ attains its minimum at a point $s^*\in(\frac{1}{5},\frac{1}{4})$, and

$$\min_{[0,1]}H=H(s^*)>\frac{8}{3\cdot(1-\frac{1}{5^2})}+\log\frac{3}{4}>2.49>\log 12,$$

proving inequality (3.8).

Making use of inequalities (3.6)–(3.8), we get

$$Q(\zeta)\leqslant\exp\Big\{\frac{8}{3(1-x^{*2})}\Big\},\quad \zeta\in\mathcal{Z}_2(f).$$

Juxtaposing this with the inequality

$$\Big|\frac{a-\zeta}{1-a\bar\zeta}\frac{a+\zeta}{1+a\bar\zeta}\Big|\leqslant\exp\Big\{-\frac{1}{3}\cdot\frac{1-a^2}{1-x^{*2}}\Big\},\quad \zeta\in\mathcal{Z}_2(f),$$

proven above, we get

$$\prod_{\zeta\in\mathcal{Z}_2(f)}Q(\zeta)\leqslant\Big(\frac{1}{|B_2(a)B_2(-a)|}\Big)^{8/(1-a^2)}.$$

Plugging this in (3.5), we obtain

$$\max_{[-a,a]} |R| \leqslant \Big(\frac{1}{|B_2(a)B_2(-a)|}\Big)^{8/(1-a^2)} \min_{[-a,a]} |R|. \tag{3.9}$$

At last, according to the classical Remez inequality (see, for example, [DR] or [BG]), for any sub-interval $I \subset [-a, a]$ and any measurable subset $E \subset I$,

$$\max_I |P| \leqslant \Big(\frac{4|I|}{|E|}\Big)^N \sup_E |P|. \tag{3.10}$$

Bringing all the information together, we obtain the estimate

$$\sup_I |f| \leqslant \Big(\frac{4|I|}{|E|}\Big)^N \exp\Big\{\frac{8}{1-a^2}\log\frac{1}{|f(a)f(-a)|}\Big\} \sup_E |f| = \Big(\frac{4|I|}{|E|}\Big)^\sigma e^{\frac{8}{3}\sigma},$$

the first factor coming from the Remez inequality (3.10) for polynomials, and the second factor being the estimate of oscillation of "almost-constant" factors $U$, $B_1$ and $R$ (correspondingly, (3.2), (3.3), and (3.9)). It remains to note that $e^8 < 4^6 = 16^3$ and $4 \cdot 16 = 64$. ∎

## 4. Proof of Lemma C

$T$ IS ONE-TO-ONE IN THE BALL $B(0, r_0)$. We show that the function $r \mapsto r\varphi(r^2)$ where, as before, $\varphi(\zeta) = (A-\zeta)/(1-A\zeta)$ is increasing on the interval $[0, r_0]$. Set $R = r^2$. We have

$$\frac{d}{dr}(r\varphi(r^2)) = \varphi(R)\Big(1 + 2R\frac{\varphi'(R)}{\varphi(R)}\Big).$$

Since $0 \leqslant R \leqslant R_0 < A$, we have $\varphi(R) > 0$. So, it will suffice to show that $|\varphi'(R)|/\varphi(R) \leqslant \frac{1}{2}$. A direct computation yields

$$\frac{|\varphi'(R)|}{\varphi(R)} \leqslant \frac{|\varphi'(R)|}{\varphi(R)^2} = \frac{1-A^2}{(A-R)^2} \leqslant \frac{2(1-A)}{(A-R_0)^2} \leqslant \frac{2\delta}{9} < \frac{1}{30},$$

since $\delta \leqslant \frac{1}{8}$. ∎

$TB(0, r_0)$ IS A BALL CENTERED AT THE ORIGIN WITH RADIUS BIGGER THAN $1 - 2\delta$. It is clear now that $TB(0, r_0) = B(0, r_0\varphi(R_0))$, so we need only to show that $r_0\varphi(R_0) > 1 - 2\delta$. We have

$$1 - AR_0 = 1 - (1-\delta^3)(1 - 3\delta - \delta^3) = 3\delta + \delta^3 + \delta^3(1 - 3\delta - \delta^3) \leqslant 3\delta + 2\delta^3$$

and, thereby,

$$\varphi(R_0) = \frac{A - R_0}{1 - AR_0} \geqslant \frac{3\delta}{3\delta + 2\delta^3} > \frac{1}{1+\delta^2}.$$

Thus, to prove our inequality, we need to check that

$$1 - 3\delta - \delta^3 \geqslant (1-2\delta)^2(1+\delta^2)^2.$$

The right-hand side does not exceed

$$(1 - 4\delta + 4\delta^2)(1 + 3\delta^2) \leqslant 1 - 4\delta + 7\delta^2.$$

Since $\delta \leqslant \frac{1}{8}$, we have $7\delta^2 + \delta^3 < 8\delta^2 \leqslant \delta$, finishing the proof. ∎

THE JACOBIAN $|\det D_x T|$ IS A LOGARITHMICALLY CONCAVE FUNCTION IN $B(0, r_0)$. First, we compute the Jacobian. Let $T_i(x) = \varphi(|x|^2)x_i$. Then

$$\frac{\partial T_i}{\partial x_j} = \begin{cases} \varphi'(r^2)2x_i x_j, & i \neq j \\ \varphi(r^2) + \varphi'(r^2)2x_i^2, & i = j \end{cases}$$

whence $\det D_x T = \det(\xi I + A)$, where $\xi = \varphi(r^2)$ and $A_{ij} = 2\varphi'(r^2)x_i x_j$. Since the rank of $A$ is one, $\det(\xi I + A) = \xi^n + \xi^{n-1}\operatorname{tr}(A)$, and

$$|\det D_x T| = (\varphi(r^2) + 2r^2\varphi'(r^2)) \cdot (\varphi(r^2))^{n-1}.$$

(This result can also be obtained in a purely geometric way: just consider the image of a small domain containing $x$ and bounded by two concentric spheres and a thin cone.)

The Taylor expansions

$$\varphi(R) = A - (1 - A^2)\sum_{k=1}^{\infty} A^{k-1}R^k$$

and

$$\varphi(R) + 2R\varphi'(R) = A - (1 - A^2)\sum_{k=1}^{\infty}(1 + 2k)A^{k-1}R^k$$

immediately show that both $\varphi(r^2)$ and $\varphi(r^2) + 2r^2\varphi'(r^2)$ are concave decreasing functions of $r$ on the interval $[0, 1]$. Since they are also positive on $[0, r_0]$, they are logarithmically concave on that interval. Hence the function $r \mapsto [\varphi(r^2) + 2r^2\varphi'(r^2)][\varphi(r^2)]^{n-1}$ is also logarithmically concave on the interval $[0, r_0]$. It remains to recall that if $\Phi(r)$ is a decreasing logarithmically concave function on the interval $[0, r_0]$, then $x \mapsto \Phi(|x|)$ is logarithmically concave in the ball $B(0, r_0) \subset \mathbb{R}^n$. ∎

THE PRE-IMAGE $T^{-1}B$ OF EVERY (REAL) BALL $B \subset TB(0,r_0)$ IS CONVEX. Since the pre-image $T^{-1}B$ is a body of revolution around the axis containing both the origin and the center of the ball $B$, it is enough to prove our statement on the plane $\mathbb{R}^2$. In order to do so, we shall show that the curvature of the image of any straight line tangent to the boundary of $T^{-1}B$ does not exceed the curvature of the boundary of $B$ which is $1/\operatorname{rad}(B)$. It is going to be a simple but somewhat boring exercise in differential geometry.

Let $rx$ ($0 \leqslant r \leqslant r_0$, $x \in \mathbb{R}^2$, $|x| = 1$) be a point on the boundary of $T^{-1}B$ and let $y(t) = rx + tv$ ($v \in \mathbb{R}^2$, $|v| = 1$, $t \in \mathbb{R}$) be the corresponding tangent line. Let $\alpha$ be the angle between the vectors $x$ and $v$. The image of our tangent line is the curve

$$\sigma(t) = \varphi(|y(t)|^2)y(t) = \varphi(r^2 + 2rt\cos\alpha + t^2)(rx + tv).$$

To estimate the curvature, we need to compute the first and second derivatives of $\sigma$. Differentiation yields

$$\begin{aligned}
\sigma'(t) &= \varphi(|y(t)|^2)v + 2\varphi'(|y(t)|^2)\langle y(t), v\rangle y(t),\\
\sigma''(t) &= 4\varphi'(|y(t)|^2)\langle y(t), v\rangle v + 2\varphi'(|y(t)|^2)y(t) + 4\varphi''(|y(t)|^2)\langle y(t), v\rangle^2 y(t).
\end{aligned}$$

Plugging in $t = 0$ and denoting, as above, $r^2 = R$, we obtain

$$\begin{aligned}
\sigma'(0) &= \varphi(R)v + 2R\varphi'(R)(\cos\alpha)x,\\
\sigma''(0) &= 4r\varphi'(R)(\cos\alpha)v + 2r\varphi'(R)x + 4rR\varphi''(R)(\cos^2\alpha)x.
\end{aligned}$$

Now we are ready to estimate the curvature. We shall use the standard formula

$$\text{curvature} = \frac{|\sigma'(0) \times \sigma''(0)|}{|\sigma'(0)|^3}.$$

We have

$$\frac{|\sigma'(0)|}{\varphi(R)} \geqslant 1 - 2R\frac{|\varphi'(R)|}{\varphi(R)}\cos\alpha \geqslant \frac{14}{15}$$

(recall that $R < 1$ and $|\varphi'(R)|/\varphi(R) \leqslant \frac{1}{30}$), and therefore $|\sigma'(0)|^3 \geqslant \frac{4}{5}\varphi(R)^3$. Using these estimates, we finally obtain

$$\begin{aligned}
\text{curvature} &\leqslant \frac{|\sigma'(0) \times \sigma''(0)|}{\frac{4}{5}\varphi(R)^3}\\
&= \frac{5}{2}r|v \times x| \cdot \Big|\frac{\varphi'(R)}{\varphi(R)^2} + 2R\Big[\frac{\varphi''(R)}{\varphi(R)^2} - \frac{2\varphi'(R)^2}{\varphi(R)^3}\Big]\cos^2\alpha\Big|\\
&= \frac{5}{2}r|\sin\alpha| \cdot \Big|\frac{\varphi'(R)}{\varphi(R)^2} + 2R\Big[\frac{\varphi'(R)}{\varphi(R)^2}\Big]'\cos^2\alpha\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{5}{2}\Big[\frac{1-A^2}{(A-R)^2} + 4R\frac{1-A^2}{(A-R)^3}\Big] \\
&\leqslant \frac{10(1-A^2)}{(A-R)^3}\cdot\Big[1+\frac{A-R}{4}\Big] \\
&\leqslant \frac{20(1-A)}{(A-R)^3}\cdot\frac{5}{4} \\
&= \frac{25}{27} < 1 < \frac{1}{\operatorname{rad}(B)},
\end{aligned}
$$

completing the proof of Lemma C. ∎

## 5. An example

Let $Q(z)$ be an arbitrary polynomial. Let $\eta > 0$ be so small that

$$\eta \max_{|z|\leqslant 1} |Q(z)| < \frac{1}{8}.$$

Consider the analytic function $F$ in the unit ball $B_c(0,1) \subset \mathbb{C}^2$ defined by

$$F(z_1, z_2) = \frac{1}{2}\Big[2\eta Q(z_1) + z_2 + \frac{1}{2}\Big]$$

and take rectangles

$$V_\delta = \Big\{0 \leqslant x_1 \leqslant \frac{1}{4},\quad 0 \leqslant x_2 + \frac{1}{2} \leqslant \delta\Big\} \subset B\Big(0, \frac{3}{4}\Big),\quad 0 < \delta \leqslant \frac{1}{2}.$$

It is easy to see that $|F| \leqslant 1$ in $B_c(0,1)$ and $|F(0,0)| \geqslant \frac{1}{8}$ regardless of the choice of $Q$. Notice that for very small $\delta > 0$, the distribution of $|F|$ in the rectangle $V_\delta$ with respect to the normalized area $\frac{1}{\operatorname{Area}(V)} d\operatorname{Area}(x)$ is practically indistinguishable from the distribution of $\eta Q(t)$ on the interval $[0, \frac{1}{4}]$ with respect to the normalized Lebesgue measure $4dt$. If the estimates (1.1) and (1.2) of the theorem were true in every rectangle $V_\delta$, they would also hold for the measures of level sets of the polynomial $\eta Q(t)$ on the interval $[0, \frac{1}{4}]$. Since they are scale-invariant, they would also hold for the measures of level sets of the polynomial $Q$ on the interval $[0, \frac{1}{4}]$. But, since polynomials are dense in the space of continuous functions, this would imply that they hold for level sets of *any* continuous function $g(t)$ on the interval $[0, \frac{1}{4}]$, which is clearly false. ∎

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